Final exam in Mathematics

GYMNASIUM MUTTENZ

Name:	Class:

The following rules apply:

- The duration of the exam is 4 hours.
- Additional Aid: English-German dictionary
- The solution process for all problems must be written down clearly and completely. Show your use of the CAS calculator[1]. The memory of the calculator has to be cleared before the exam.
- The exam consists of two parts:
  - Part 1: Solve problems 1 to 3 with the aid of the mathematics formulary[2] and a simple calculator[3].

    Once you are finished solving this part, place all corresponding pages (including the exercise sheets) into the envelope provided. Seal the envelope and hand it in to the supervisor. (Attention: Only the pages in the sealed envelope will be considered for the assessment of part 1!)
  - Part 2: Once you hand in part 1 in a sealed envelope, you will receive your CAS calculator. Solve problems 4 to 7 with the aid of your CAS calculator and the mathematics formulary.
- The final grade is calculated as follows:

$$\text{Final grade} = \frac{5 \cdot \text{“achieved points”}}{42} + 1 \text{ (rounded to half a mark)} = \boxed{}$$

We wish you much success!

Problem	1	2	3	4	5	6	7	Total
Possible Points	4.5	7.5	8	9	5	5.5	10	49.5
Achieved Points								

[1] TI-Nspire CX-T II CAS
[2] Adrian Wetzel. *Formelsammlung Mathematik*. 9th ed. 2021. ISBN: 978-3-9523907-1-9.
[3] TI-30 ECO RS, TI-30X A, TI-30Xa Solar oder TI-30X IIS

Name:	Class:

Part 1: Without CAS calculator

Figure 1: Without CAS calculator[4]

The problems in this part have to be solved without the CAS calculator. The aids allowed in this part of the exam are the mathematics formulary[5] and a simple calculator[6].

[4]Wikimedia Commons. *TI-Nspire CX-T CAS II.*. Creative Commons Attribution-Share Alike 4.0 International. URL: `https://commons.wikimedia.org/wiki/File:TI-Nspire_CX-T_CAS_II.jpg` (visited on 02/05/2024)

[5]Adrian Wetzel. *Mathematics formulary*. 4th ed. 2021. ISBN: 978-3-9523907-2-6.

[6]TI-30 ECO RS, TI-30X A, TI-30Xa Solar oder TI-30X IIS

Problem 1 (4.5 P)

Given are the three vectors:

$$\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} 2 \\ 1 \\ z \end{pmatrix} \text{ and } \vec{c} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

(a) Calculate the length of $\vec{a}$. (1 P)

(b) The vectors $\vec{a}$ and $\vec{b}$ are perpendicular. Calculate the component z. (1.5 P)

(c) Calculate the area of the parallelogram spanned by the vectors $\vec{a}$ and $\vec{c}$. (2 P)

Problem 1: achieved points: of 4.5 P

Problem 2 (7.5 P)

Of lines g and h we know the following:

- line g has direction vector $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and passes through the origin.
- line h has direction vector $\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and intersects line g in point $P = (4,\, y_P,\, z_P)$.

(a) Determine a parametric equation of line g. (1 P)

(b) Determine a parametric equation of line h. (1 P)

(c) The lines g and h span a plane. Calculate the distance between this plane and point $Q = (0,\, 2,\, 1)$. (3 P)

(d) Give an example of a line j that is parallel to h and skew to line g. Show mathematically that j and g are indeed skew. (2.5 P)

Problem 2: achieved points: of 7.5 P

Problem 3 (8 P)

For the next 16 statements you need to decide if the statement is TRUE or FALSE.

This problem is graded as follows:

- Correct answer: +0.5 P
- The first four incorrect answers: 0 P
- Further incorrect answers: −0.5 P
- No answer: 0 P

Minimum possible points: 0 P.

(a) *Vector Geometry*

(i) The line

$$g:\ \vec{r} = \begin{pmatrix} 2024 \\ 2024 \\ 2024 \end{pmatrix} + t \cdot \begin{pmatrix} 2024 \\ 0 \\ 2024 \end{pmatrix}$$

intersects the plane with the equation

$$x + 2024 \cdot y - z = 0.$$

true ☐ false ☐ (0.5 P)

(ii) For two vectors $\vec{v}$ and $\vec{w}$ in in three dimensional space the following is always true:

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = 0.$$

true ☐ false ☐ (0.5 P)

(iii) From $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}|$ it follows that $|\vec{v} + \vec{w}| = |\vec{v}| + |\vec{w}|$.

true ☐ false ☐ (0.5 P)

(b) *Probability and Combinatorics*

(i)

$$\binom{1000}{500} = \frac{1000!}{(500!)^2}$$

true ☐ false ☐ (0.5 P)

(ii)

$$99! = \frac{100!}{2}$$

true ☐ false ☐ (0.5 P)

(iii) The number of possibilities to receive exactly three consecutive "heads" in 8 throws of a fair coin is $\binom{8}{3}$.

true ☐ false ☐ (0.5 P)

(iv) A die is thrown 2024 times. The probability of never throwing a ⚅ is strictly larger than 0.

true ☐ false ☐ (0.5 P)

(c) *Calculus*

(i) $$\int_{-2024}^{2024} x^2 \,\mathrm{d}x > 0$$ true ☐ false ☐ (0.5 P)

(ii) $$\int_{-2024}^{\sqrt{2024}} x \,\mathrm{d}x > 0$$ true ☐ false ☐ (0.5 P)

(iii) The shaded area in Figure 2 can be calculated by

$$\left|\int_{-a}^{a} f(x)\,\mathrm{d}x\right|.$$

true ☐ false ☐ (0.5 P)

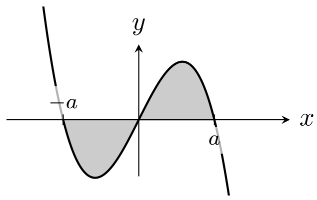


Figure 2: graph of function f and shaded area

(iv) The function $f(x) = -6 - x^2$ has more than one zero. true ☐ false ☐ (0.5 P)

(v) The graph of a polynomial function always has more extrema than points of inflection. true ☐ false ☐ (0.5 P)

(vi) The graph of a polynomial function of degree five can have four points of inflection. true ☐ false ☐ (0.5 P)

(vii) There is a function F that is the antiderivative of $f(x) = x$ as well as of $g(x) = x + 1$. true ☐ false ☐ (0.5 P)

(viii) If $f(x) = \sin(2 \cdot x)$, then $f'(5 \cdot \pi) < 0$. true ☐ false ☐ (0.5 P)

(ix) The 2024th derivative of $f(x) = \sin(2 \cdot x)$ is $2^{2024} \cdot f(x)$. true ☐ false ☐ (0.5 P)

Problem 3: achieved points: ☐ of 8 P

Name:	Class:

Part 2: With CAS calculator

Figure 3: With CAS calculator[7]

Once you hand in part 1 in a sealed envelope you will receive your CAS calculator. Solve problems 4 to 7 with the aid of your CAS calculator[8] and the mathematics formulary.

[7]Wikimedia Commons. *TI-Nspire CX-T CAS II.*. Creative Commons Attribution-Share Alike 4.0 International. URL: `https://commons.wikimedia.org/wiki/File:TI-Nspire_CX-T_CAS_II.jpg` (visited on 02/05/2024)

[8]TI-Nspire CX-T II CAS

Problem 4 (9 P)

Circle K passes through the origin $O = (0, 0)$ and its centre M is $M = (4, 3)$.

The graph of the parabola

$$p(x) = a \cdot x^2 + b \cdot x + c$$

intersects the circle in the origin as well as in point $P = (x_P, 6)$. P is a maximum of the graph of p.

(a) Determine the equation of line g through points O and M. (0.5 P)

(b) Determine the equation of p. (3 P)

Hint: If you cannot solve (b) continue with the function

$$p(x) = -\frac{3}{32} \cdot x^2 + \frac{3}{2} \cdot x$$

.

(c) Determine the equation of the tangent t to the graph of p that is parallel to line g. (2.5 P)

(d) Calculate the shaded area in Figure 4. (3 P)

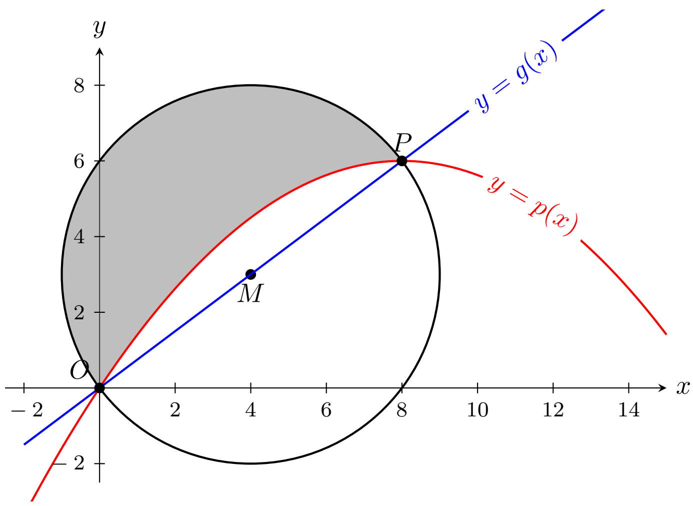
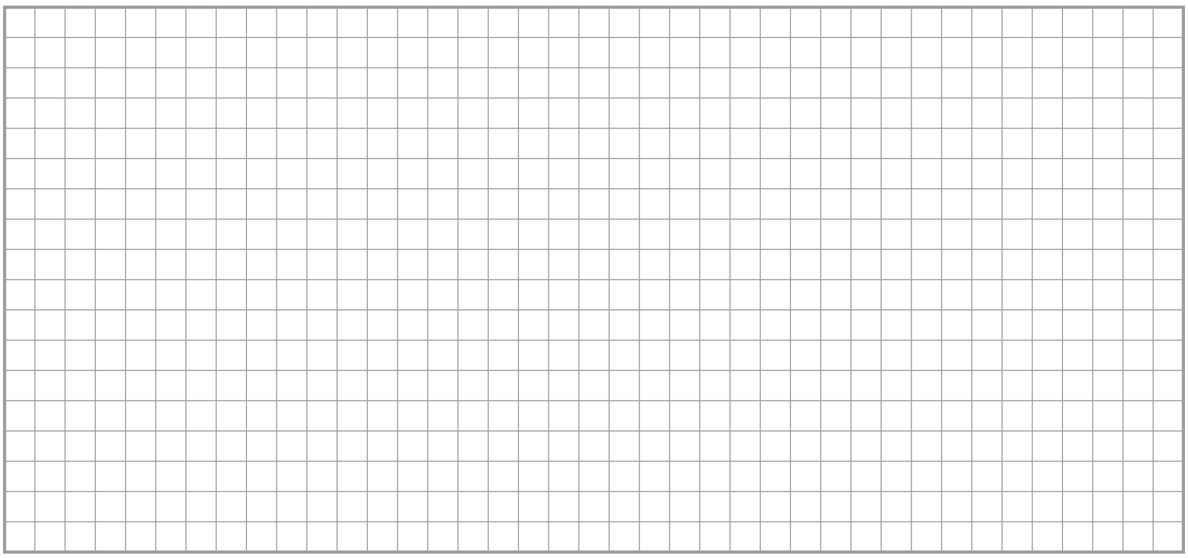


Figure 4: area



Problem 4: achieved points: of 9 P

Problem 5 (5 P)

Starting in point $P = (0,\, 2)$ an infinite sequence of lines is drawn, as shown in Figure 5. Each line is perpendicular to the previous line. The lengths of the lines form a geometric sequence.

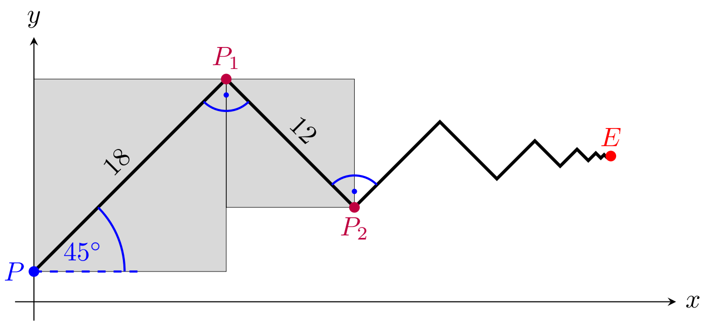


Figure 5: sketch to Problem 5

(a) Calculate the x-coordinates of points P_1 and P_2. (1.5 P)

(b) Calculate the coordinates of the "ending point" E. (3.5 P)

Problem 5: achieved points: ☐ of 5 P

Problem 6 (5.5 P)

For his final project as a chocolatier the apprentice Julian wants to create a very special new praline:

The confectionary is to have the form of straight pyramid with a quadratic base. What makes the praline special is the icing with gold powder that covers the visible surface of the delicacy. The praline is to have a volume of $12\,\text{cm}^3$. The special gold icing is so thin that it does not need to be taken into consideration when calculating the volume.

However, the cost of the icing is at $1.00\,\frac{\text{CHF}}{\text{cm}^2}$ rather pricey. For the underside of the pyramid a similar icing with less gold powder is used. It costs $0.80\,\frac{\text{CHF}}{\text{cm}^2}$.

What height of the pyramid should Julian choose if the praline is to be produced as cheaply as possible? (5.5 P)

The proof of minimum is not necessary.

Problem 6: achieved points: | | of 5.5 P

Problem 7 (10 P)

A gambling machine has three identical wheels. Each of these wheels only contains the four symbols ★, ✠, ♦ and ●. If you toss in a coin the wheels start to turn and then, independently from one another, stop at random . When still, each wheel shows exactly one image.

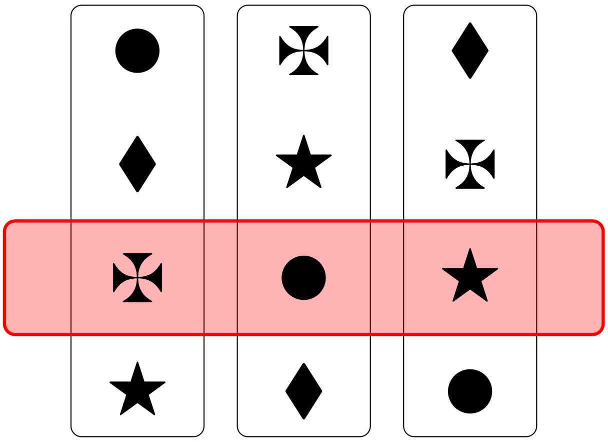


Figure 6: wheels

- To play you pay 1 CHF.
- If exactly two wheels show the same symbol, the payout is 2 CHF.
- If all three wheels show the same symbol, the payout differs:
  - If the wheels show three ★, the payout is 20 CHF.
  - If the wheels show three ✠, the payout is 10 CHF.
  - If the wheels show three ♦ or three ●, the payout is 5 CHF.
- If the wheels show three different symbols, you lose your stake.

(a) How many possibilities are there for arranging the symbols if the order matters? (1 P)

(b) How many possibilities are there for arranging the symbols if the order does not matter? (1.5 P)

(c) How many possibilities are there for arranging the symbols if the order does not matter and one of the symbols has to appear exactly twice? (1 P)

(d) What is the probability of losing a game? (1 P)

(e) What is the probability that a player wins at least once in five games? (1 P)

(f) What is the probability that a player wins 20 CHF exactly twice and loses the other games? (2 P)

(g) What kind of win or loss can a player expect in this game? (2.5 P)

Problem 7: achieved points: | | of 10 P