Gymnasium Muttenz Final Exam 2016
Mathematics Profiles A and B

Name:...	Class:...............

Information:

- The exam lasts 4 hours.
- There are 48 points to be made.
- All problems will be evaluated.
- Show your thinking in all steps. Declare the use of your CAS calculator.

Process:

- **Part A:** You receive problems (1), (2) and (3), which are to be solved using your CAS calculator and the formula book (FS A. Wetzel).
- **Teil B:** After you hand in your calculator, you will receive problems (4) and (5), which must be solved with only the aid of the formula book. You may continue working on problems (1), (2) and (3) without your calculator. Hand in your solutions to all problems at the end of the exam.

Assessment:

Problem	**Points made**
(1) Calculus 4P / Conic Sections 3P / Sequences & Series 3P	
(2) Calculus & Complex Numbers 10P	
(3) Probability & Statistics 10P	
(4) Vector Geometry 7P / Calculus 3P	
(5) All topics 8P	
Total points made:	

$\text{Grade} = \frac{\text{Total points made}\cdot 5}{40} + 1$ rounded to half a mark **Grade:**	

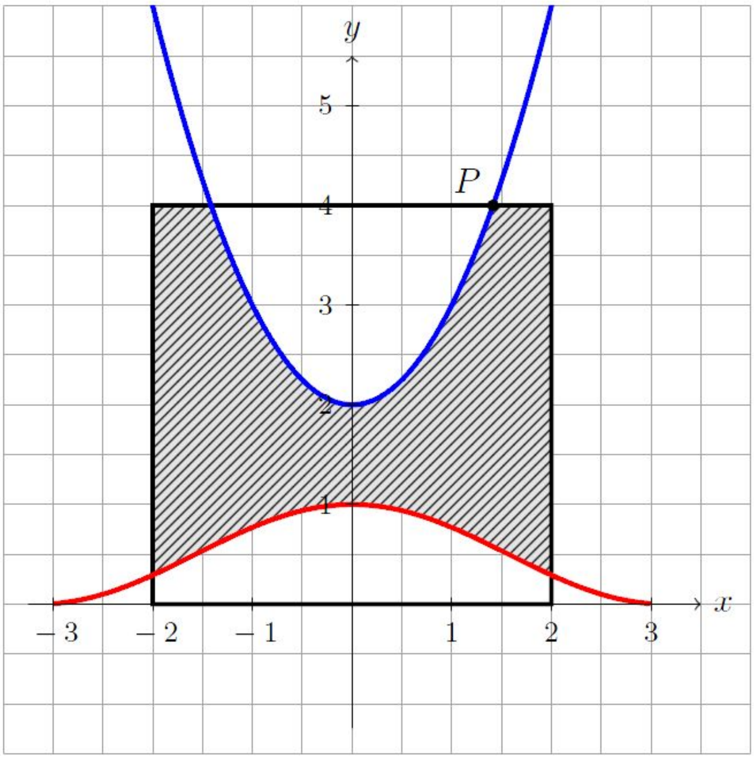
Part A with calculator

Problem 1 (10 points)

Short Problem 1.1

Given the graphs of the functions f and g.

$$f : y(x) = x^2 + 2 \text{ and } g : y(x) = \frac{1}{2} \cdot \cos(x) + \frac{1}{2}$$



(a) Calculate the acute angle α between the tangent to f in $P = (x_P, 4)$ (in the first quadrant) and the x-axis. [1.5P]

(b) The graphs of f and g define the shaded area in the square with corners $(-2, 0)$, $(2, 0)$, $(2, 4)$ und $(-2, 4)$. Calculate this area. [2.5P]

Short Problem 1.2

The ellipse given by the equation $3x^2 + 5y^2 = 120$ and a hyperbola H have common foci and the common point $P = (5, 3)$. Determine an equation for H. [3P]

Short Problem 1.3

Given an infinite number of circles, whose radii form a decreasing geometric sequence. The sum of the radii of the two largest circles is equal to the sum of the remaining radii. Determine the quotient q of this geometric sequence. [3P]

Problem 2 (10 points)

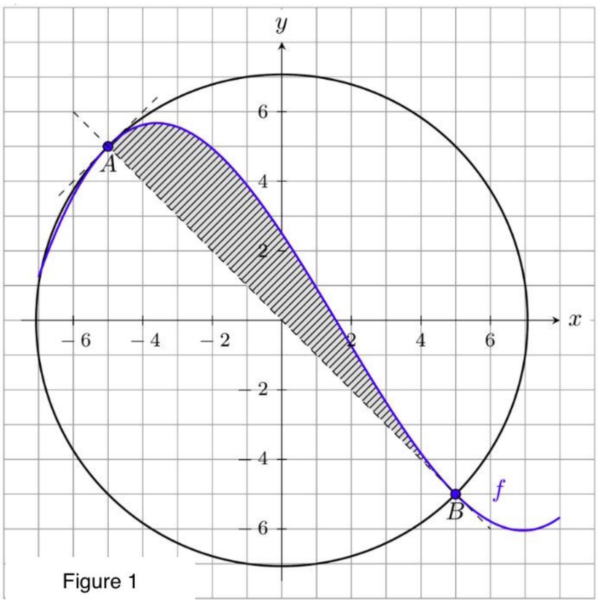
A circle with midpoint $M = (0, 0)$ passes through $A = (-5, 5)$ and $B = (5, -5)$. Shown further is the graph of a polynomial of degree three f, tangent to the circle in A and intersecting the circle at a right angle in B (Figure 1).

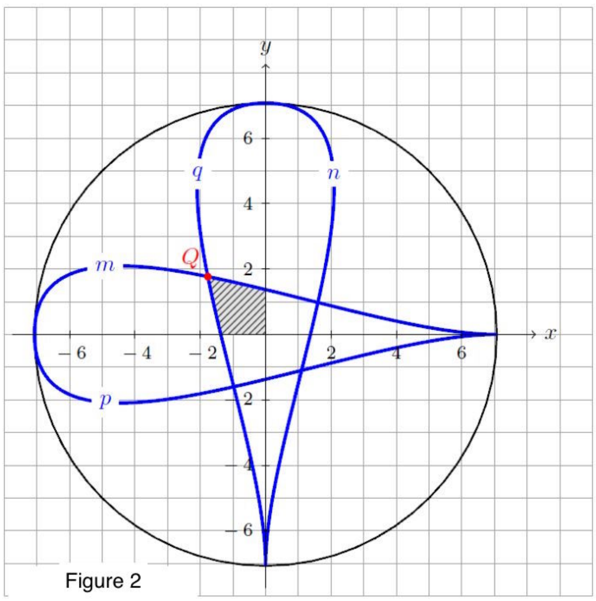
(a) Determine the equation of f. [2P]

Note: If you could not solve (a), use $f : y(x) = \frac{1}{50} \cdot x^3 - \frac{1}{10} \cdot x^2 - \frac{3}{2} \cdot x + \frac{5}{2}$.

(b) Calculate the shaded area in Figure 1. [1P]

(c) Calculate the coordinates of the point on the graph of f closest to the origin. [2P]





The curves m and n are created by rotating the graph of f by 45° and −45° respectively. Mirroring m and n across the $x-$ and $y-$axes respectively, we get the curves p and q (Figure 2).

(d) Give a complex-valued function for m. [1P]

(e) Calculate the coordinates of the point of intersection Q. [2P]

(f) The coordinate axes, m and q bound the shaded area in the second quadrant (Figure 2). Calculate its area. [2P]

Problem 3 (10 Points)

The two wheels of a bike are each locked with a combination lock. Each combination lock has a five digit code, that does not begin with a zero.

(a) Leo Larcenist assumes that the code of the first lock is an even number and that the code of the second lock consists of unique digits. If his assumption is correct: How many codes must he try at most in order to open both locks? [1.5P]

(b) Let the codes now be two random five digit numbers with no further information. What is the probability that exactly one of the positions match? [2P]

(c) If Leo Larcenist tries to steal a bike, he has a success rate of $p = 0.3$. In one night he tries to steal 10 bikes. What is the probability that he can steal more than half of them? [1.5P]

(d) If Ronny Robber tries to steal a bike, he is slightly less successful than Leo Larcenist and only has a success rate of $p = 0.2$. In one night Leo Larcenist first tries to steal 10 bikes. Afterwards, Ronny Robber attempts to steal the bikes not stolen by Leo Larcenist. What is the probability that exactly one bike is stolen on that night? [2.5P]

(e) In a city all bikes have a single combination lock. 40% of the red bikes have an even code. Of the other bikes only 30% have an even code. The following is known: If a random bike in this city has an even code, then the probability that it is red is $p = \frac{1009}{2016}$. Determine the proportion of red bikes in the city. [2.5 Punkte]

Part B without calculator

Problem 4 (10 Points)

Short Problem 4.1

Given the point $M = (2, 4, 1)$ and the line g

$$g : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -6 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

(a) Determine a coordinate equation of the plane E, such that g is perpendicular to E and M lies on E. [1P]

(b) A square with midpoint M has one corner on the line g and lies on the plane E. Calculate the coordinates of two corners of the square. [2P]

Short Problem 4.2

The point $P = (7, 1, 5)$ lies on a sphere K with midpoint $M = (3, 1, 2)$.
Further given is the plane $E_c : 2x + 3y + 6z - 7c = 0$ with $c \in \mathbb{R}$.

(a) Show that the point $A = (3, 10, 14)$ lies outside the sphere K. [1P]

(b) For which values of c do the sphere K and the plane E_c touch? [3P]

Short Problem 4.3

Determine all values of a, such that:

$$\int_0^a (1 + 2016 \cdot x)dx = 2016 \cdot a^2 \text{ [3P]}$$

Problem 5 (8 Points)

Decide in each case if the statement is true or false

Assessment:

- Every correct answer gives 0.5 points.
- The first 4 incorrect answers will not be penalized.
- Further incorrect answers are penalized with a deduction of 0.5 points.
- No answer is 0 points.
- The minimum points possible for this problem is 0.

(a) The vector $\vec{v}$ is perpendicular to the plane E. ☐ true ☐ false

$\vec{v} = \begin{pmatrix} 14 \\ -7 \\ 22 \end{pmatrix}$ $E\colon -2 \cdot x + y - 3 \cdot z + 4 = 0$

(b) $\int_{\pi}^{4\pi} \sin(x) dx < 0$ ☐ true ☐ false

(c) $\vec{a} \cdot \vec{b} = 0$ and $\vec{b} \cdot \vec{c} = 0$ implies $\vec{a} \cdot \vec{c} = 0$ ☐ true ☐ false

(d) $\sqrt{2016} \notin \mathbb{Q}$ ☐ true ☐ false

(e) $\sqrt[10]{10} > \sqrt[3]{2}$ ☐ true ☐ false

(f) Given the polynomial $f(x)$. If $f(2) \cdot f'(2) = 0$, then the graph of f has a maximum or minimum point at $x = 2$. ☐ true ☐ false

(g) The graph of a polynomial of degree 2016 has at least one point with slope 0. ☐ true ☐ false

(h) Given an arithmetic sequence with $a_1 = 2016$ where d is an odd number, then a_{2016} is also an odd number. ☐ true ☐ false

(i) Given a geometric sequence with $a_1 = 2016$ where q is an odd number, then a_{2016} is also an odd number. ☐ true ☐ false

The problems continue on the next page!

(j) $-2016 \cdot |z| \leq 0$ for every complex number z. ☐ true ☐ false

(k) $Re(z+1) = Im(z+i)$ for every complex number z. ☐ true ☐ false

(l) $Im(z+i) = Im(z) + 1$ for every complex number z. ☐ true ☐ false

(m) $i^{2016} = i^{4444}$ ☐ true ☐ false

(n) The graph of the function $f(t) = (t + i \cdot t^2) \cdot e^{i\frac{\pi}{4}}$ corresponds to a parabola that is rotated 45° around the origin. ☐ true ☐ false

(o) The graph of the function $f(t) = (t+i \cdot t^2) \cdot e^{i\frac{2\pi}{6}} \cdot (\cos(\frac{2\pi}{6}) + i \cdot \sin(\frac{2\pi}{6}))$ corresponds to a parabola that is rotated 60° around the origin. ☐ true ☐ false

(p) The equation $x^2 + x + (a \cdot y)^2 + y - 12 = 0$ describes an ellipse for all real values of a. ☐ true ☐ false